Neutrino oscillations in matter - the MSW triangle

Bo Tranberg
bo@tberg.dk

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1 Neutrino oscillations in vacuum

A neutrino of the generation α propagating through vacuum is, after a time interval t, given by

$$|\nu^\alpha\rangle_t = \sum_i U_{\alpha i} e^{-iE_i t} |\nu'^i\rangle . \tag{1}$$

From [FY] (8.3)[1] we have the relation

$$\nu^\beta = U_{\beta i}\nu'^i \Rightarrow \nu'^i = (U^\dagger)_{i\beta}\nu^\beta .$$

Inserting this relation into (1) gives a relation between generations α and β

$$|\nu^\alpha\rangle_t = \sum U_{\alpha i} e^{-iE_i t} (U^\dagger)_{i\beta} |\nu^\beta\rangle . \tag{2}$$

We only consider two generations of neutrinos giving the mixing matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{3}$$

and assume the ultrarelativistic limit

$$E_i \simeq p + \frac{m_i^2}{2E}. \tag{4}$$

Another way of writing (2) is then

$$|\nu^\alpha\rangle_t = U \begin{bmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{bmatrix} U^\dagger |\nu^\beta\rangle . \tag{5}$$

Using (4) on (5) and writing out the matrix multiplications with (3) gives

$$\begin{aligned} |\nu^\alpha\rangle_t &= e^{-ipt} U \begin{bmatrix} e^{-i\frac{m_1^2 t}{2E}} & 0 \\ 0 & e^{-i\frac{m_2^2 t}{2E}} \end{bmatrix} U^\dagger |\nu^\beta\rangle \\ &= e^{-ipt} \begin{bmatrix} \cos^2\theta e^{-i\frac{m_1^2 t}{2E}} + \sin^2\theta e^{-i\frac{m_2^2 t}{2E}} & \left(e^{-i\frac{m_2^2 t}{2E}} - e^{-i\frac{m_1^2 t}{2E}}\right)\cos\theta\sin\theta \\ \left(e^{-i\frac{m_2^2 t}{2E}} - e^{-i\frac{m_1^2 t}{2E}}\right)\cos\theta\sin\theta & \sin^2\theta e^{-i\frac{m_1^2 t}{2E}} + \cos^2\theta e^{-i\frac{m_2^2 t}{2E}} \end{bmatrix} |\nu^\beta\rangle . \end{aligned} \tag{6}$$

[1] [FY] (x) refers to equation x in *Physics of Neutrinos* by Fukugita and Yanagida.

To simplify this expression we need the following trigonometric identities.

$$\boxed{\begin{aligned} e^{-ia} - e^{-ib} &= -2i \sin\left(\frac{a-b}{2}\right) e^{-i\frac{a+b}{2}} \\ e^{-ia} + e^{-ib} &= 2e^{-i\frac{a+b}{2}} \cos\left(\frac{b-a}{2}\right) \\ \cos^2\theta &= \frac{1+\cos 2\theta}{2} \\ \sin^2\theta &= \frac{1-\cos 2\theta}{2} \\ \sin\theta\cos\theta &= \frac{\sin 2\theta}{2} \end{aligned}} \tag{7}$$

Looking at expression (6) we see that the two off-diagonal terms are identical and that there is some similarity between the two diagonal terms. We simplify the two off-diagonal terms in (6) by applying the first and the last of the relations in (7), so

$$\left(e^{-i\frac{m_2^2 t}{2E}} - e^{-i\frac{m_1^2 t}{2E}}\right)\cos\theta\sin\theta = -i\sin\frac{\Delta m^2 t}{4E}\sin 2\theta e^{-i\frac{m_1^2+m_2^2}{4E}t}. \tag{8}$$

We now look at the diagonal terms of (6) and apply relations 2, 3 and 4 from (7), so

$$\begin{aligned} \begin{pmatrix}\cos^2\theta \\ \sin^2\theta\end{pmatrix} e^{-i\frac{m_1^2 t}{2E}} + \begin{pmatrix}\sin^2\theta \\ \cos^2\theta\end{pmatrix} e^{-i\frac{m_2^2 t}{2E}} &= \frac{1\pm\cos 2\theta}{2} e^{-i\frac{m_1^2 t}{2E}} + \frac{1\mp\cos 2\theta}{2} e^{-i\frac{m_2^2 t}{2E}} \\ &= \frac{1}{2}\left(e^{-i\frac{m_1^2 t}{2E}} + e^{-i\frac{m_2^2 t}{2E}}\right) \mp \frac{1}{2}\cos 2\theta\left(e^{-i\frac{m_2^2 t}{2E}} - e^{-i\frac{m_1^2 t}{2E}}\right) \\ &= e^{-i\frac{m_1^2+m_2^2}{4E}t}\cos\frac{\Delta m^2 t}{4E} \pm i\sin\frac{\Delta m^2}{4E} e^{-i\frac{m_1^2+m_2^2}{4E}t}\cos 2\theta. \end{aligned} \tag{9}$$

Where the components on the left and the $\pm$ on the right corresponds to position (1,1) or (2,2) in (6) respectively.
Putting (9) and (8) into (6) (2) now takes the form

$$|\nu^\alpha\rangle_t = \begin{pmatrix} \cos\frac{\Delta m^2 t}{4E} + i\sin\frac{\Delta m^2}{4E}\cos 2\theta & -i\sin\frac{\Delta m^2 t}{4E}\sin 2\theta \\ -i\sin\frac{\Delta m^2 t}{4E}\sin 2\theta & \cos\frac{\Delta m^2 t}{4E} - i\sin\frac{\Delta m^2}{4E}\cos 2\theta \end{pmatrix} |\nu^\beta\rangle \tag{10}$$

omitting the common phase factor of $\exp(-i\frac{m_1^2+m_2^2}{4E}t)$. This is the same form as in [FY] (8.54). The only difference is the sign on the imaginary part of the diagonal terms.

The survival probability for a monoenergetic neutrino beam a distance L from the source is

$$P_{\nu_e\to\nu_e} = 1 - \sin^2 2\theta \sin^2\left(\pi\frac{L}{l_0}\right). \tag{11}$$

In the limit where $L \gg l_0$ the second sine oscillates rapidly and since we are at a great distance from the source it can be approximated by its time averaged value

$$\sin^2\left(\pi\frac{L}{l_0}\right) \simeq \frac{1}{2} \tag{12}$$

leaving us with

$$P_{\nu_e\to\nu_e} = 1 - \frac{1}{2}\sin^2 2\theta.$$

This can be simplified to the inequality relation

$$P_{\nu_e \to \nu_e} \geq \frac{1}{2}$$

yielding that electron neutrinos are favored over muon neutrinos a far distance from the source due to the lower mass of the electron generation.

Monoenergetic electron neutrinos emitted from a source a distance L away has the transition probability given in (11). When electron neutrinos are emitted with an energy distribution $f(E)$ rather than monoenergetic we have to adjust the part of (11) that is energy dependent. The oscillation length l_0 is dependent on the individual energy of the neutrinos. From [FY] (8.56) and [FY] (8.58) we get

$$\frac{\Delta m^2}{4E}\frac{L}{c}\frac{l_0}{l_0} = \frac{E_0}{E}\frac{L}{l_0}\pi$$

where E is a variable energy in the distribution $f(E)$ and E_0 is the mean energy of that distribution. This makes the survival probability into an integral equation over the energy distribution $f(E)$,

$$P_{\nu_e \to \nu_e} = \int_0^\infty \left[1 - \sin^2 2\theta \sin^2\left(\pi \frac{E_0}{E}\frac{L}{l_0}\right)\right] f(E) dE. \tag{13}$$

We have the Gaussian distribution

$$f(E) = Ke^{-(E-E_0)^2/\sigma^2} \tag{14}$$

where $\sigma = 0.01E$ and

$$K = \frac{1}{\int_0^\infty e^{-(E-E_0)^2/\sigma^2} dE}.$$

Inserting these in (13) gives the survival probability

$$P_{\nu_e \to \nu_e} = \int_0^\infty \left[1 - \sin^2 2\theta \sin^2\left(\pi \frac{E_0}{E}\frac{L}{l_0}\right)\right] Ke^{-\frac{E-E_0}{\sigma^2}} dE$$

This integral has been plotted in Figure (1) with the values $\Delta m^2 = 7.59 \times 10^{-5}\,\text{eV}^2$ and $\sin^2\theta = 0.861$ [2]. It is evident that the electron eigenstate is favoured over the muon eigenstate. For one the survival probability never equals zero and for large values of L/l_0, $P_{\nu_e \to \nu_e}$ converges to 0.5695. The same value can be obtained if the integral is solved analytically under the approximation (12).

[2] https://en.wikipedia.org/wiki/Neutrino_oscillations

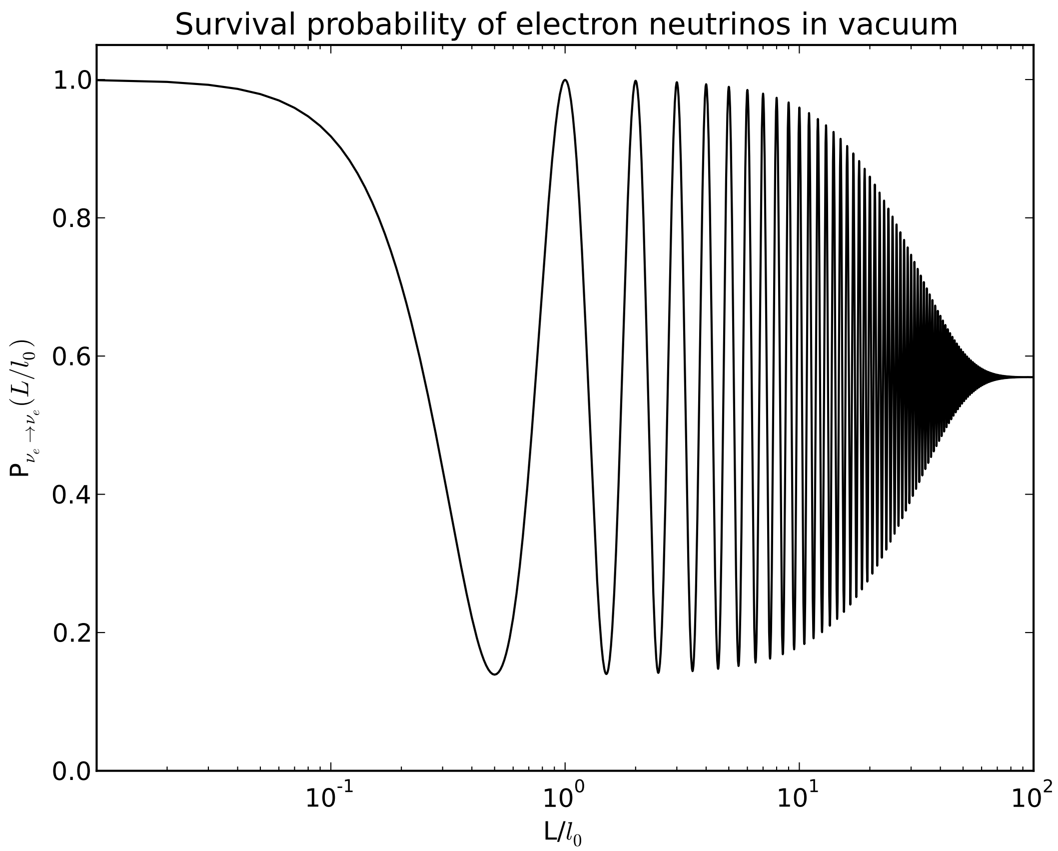

Figure 1: Survival probability of electron neutrinos in vacuum with energy distribution (14) as a function of L/l_0.

2 Neutrino oscillations in matter

For neutrino oscillations in matter we have the mixing parameter

$$A = 2\sqrt{2}G_F E n_e. \tag{15}$$

A resonance between the two eigenstates in matter occurs when $A/\Delta m^2 = \cos 2\theta$, that is to say the electron density, n_e, must equal the critical electron density,

$$n_e = n_{e,crit} \equiv \frac{1}{2\sqrt{2}G_F}\frac{\Delta m^2}{E}\cos 2\theta. \tag{16}$$

In the case where $\Delta m^2 < 0$ the critical electron density (16) would become negative making the resonance impossible for neutrinos. From [FY] (3.239) we know that the sign on (15) would be negative if we are dealing with anti neutrinos. For anti neutrinos the resonance therefore only occurs if $\Delta m^2 < 0$.

The general survival probability of electron neutrinos in matter is

$$P_{\nu_e \to \nu_e} = \frac{1}{2} + \left(\frac{1}{2} - P_f\right)\cos 2\theta \cos 2\tilde{\theta} \tag{17}$$

with

$$P_f = e^{-\frac{\pi}{2}\gamma} \tag{18}$$

and

$$\gamma = \frac{\Delta m^2 \sin^2 2\theta}{2E\cos 2\theta (1/n_e)(dn_e/dr)}. \tag{19}$$

The resonance discussed in Section 2.1 occurs in the adiabatic region where the derivative $(1/n_e)(dn_e/dr)$ is small. In that case (19) gets big making (18) very small and thereby simplifying (17). We consider the general case of (17) where

$$\cos 2\tilde{\theta} = \frac{-A/\Delta m^2 + \cos 2\theta}{\sqrt{(A/\Delta m^2 - \cos 2\theta)^2 + \sin^2 2\theta}}.$$

We rewrite (19) with the parameters

$$\begin{aligned} x &= \frac{2E}{\Delta m^2}\sqrt{2}G_F n_e = \frac{A}{\Delta m^2} \\ &= \frac{E}{1\,\text{MeV}}\frac{1.52\times 10^{-5}\,\text{eV}^2}{\Delta m^2} \end{aligned} \tag{20}$$

and

$$\frac{1}{n_e}\frac{dn_e}{dr} = 10.5 R_\odot$$

giving a compact expression for γ,

$$\begin{aligned} \gamma &= \frac{\Delta m^2}{2E}\frac{R_\odot}{10.5}\frac{1}{x}\frac{E}{1\,\text{MeV}}\frac{1.52\times 10^{-5}\,\text{eV}^2}{\Delta m^2}\frac{1}{\hbar c} \\ &= 2.554\times 10^3\,\frac{\sin^2 2\theta}{\cos 2\theta}\frac{1}{x} \end{aligned} \tag{21}$$

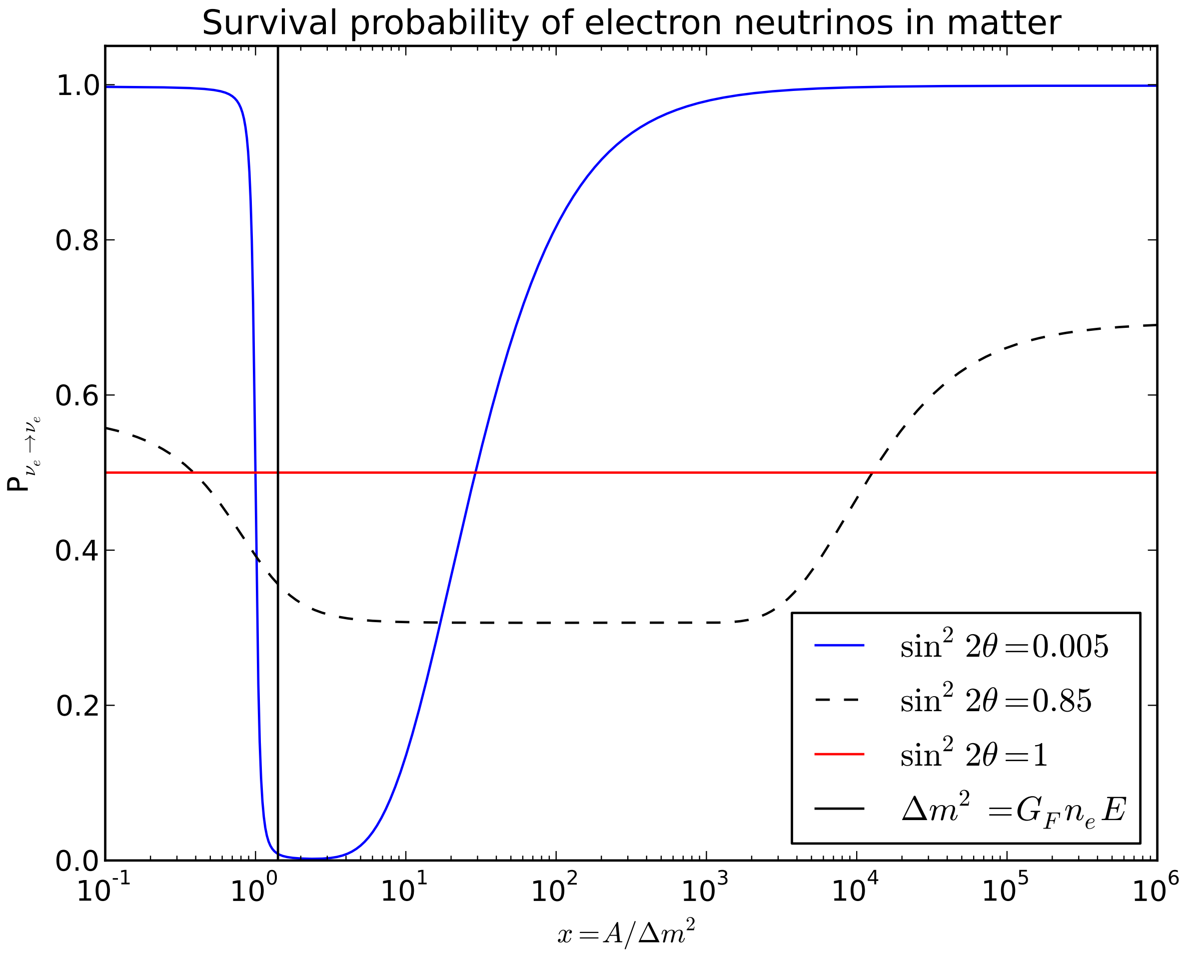

Figure 2: Survival probability of electron neutrinos in matter as a function of $A/\Delta m^2$ for different values of $\sin^2 2\theta$.

Figure (2) shows the survival probability of electron neutrinos in matter (17) plotted as a function of $x = A/\Delta m^2$ (20) for three values of $\sin^2 2\theta$.

Figure 3 shows my interpretation of the MSW triangle in [FY] (Figure 8.10). The color scale indicates values of the survival probability (17) as a function of $x = A/\Delta m^2$ and $\sin^2 2\theta$. In short blue means $\nu_e \to \nu_\mu$ and red means $\nu_e \to \nu_e$.

The vertical line of constant Δm^2 in Figure 2 corresponds to the "top" of the triangle in Figure 3 (imagine a horizontal line) with

$$\Delta m^2 = G_F n_e E \Rightarrow x = \sqrt{2}. \tag{22}$$

comparing the two figures it is seen that for Δm^2 larger than this value (smaller x) the survival probability is higher since the conversion only happens inside the triangle in Figure 3, for small mixing angles at least. Increasing x in Figure 2 corresponds to starting at the top of Figure 3 and moving vertically down at constant z.

The blue line in Figure 2 represents the survival probability for electron neutrinos at a small mixing angle, θ (in the adiabatic region), that is to the left of Figure 3. The dashed line in Figure 2 corresponds to a larger value of θ close to the limit of the adiabatic condition that is required for the resonance to occur.

The small value of θ (blue line) is well inside the adiabatic region. This is why the conversion $\nu_e \to \nu_\mu$ only happens in the region corresponding to the inside of the triangle in Figure 3 giving

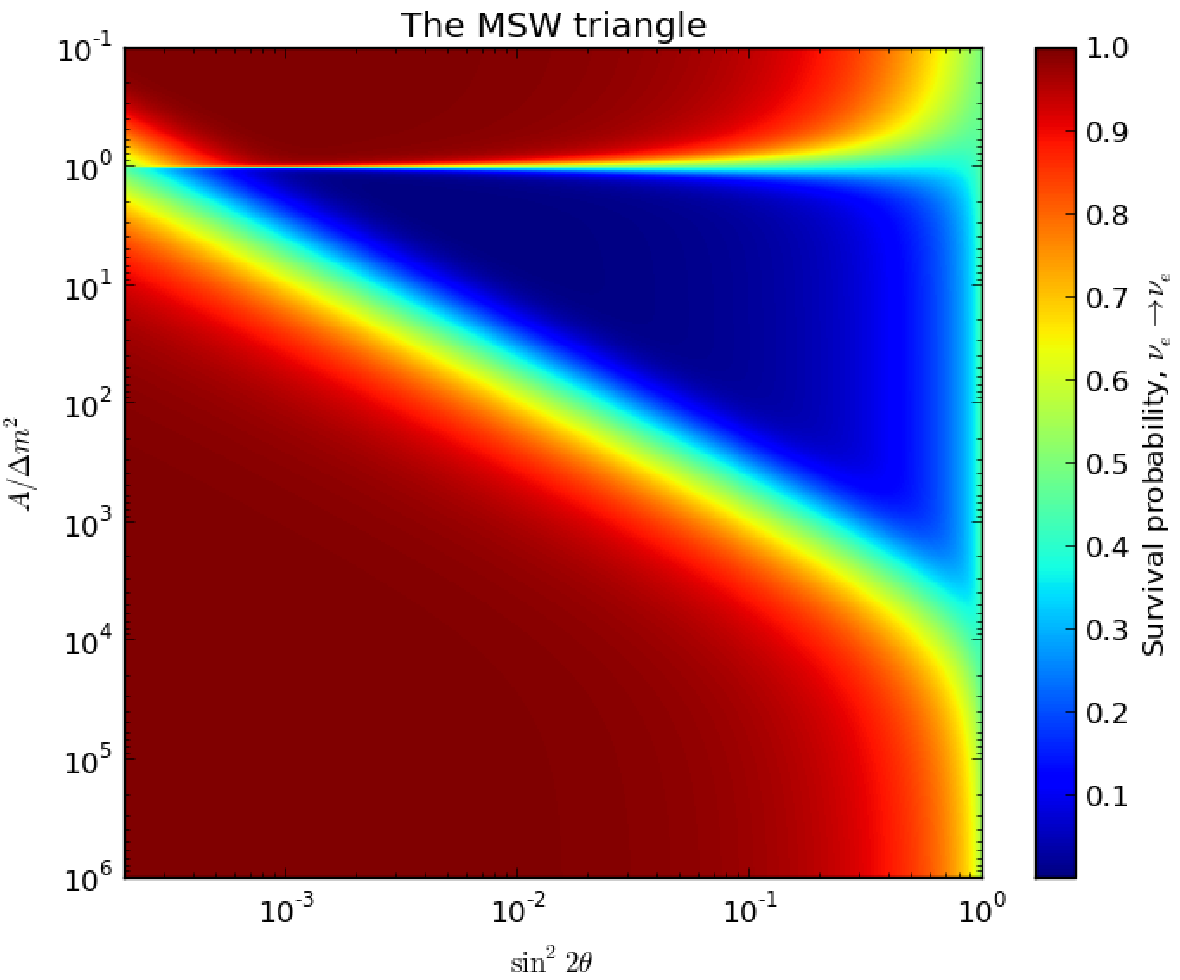

Figure 3: The MSW triangle. The color scale represents values of the survival probability $P_{\nu_e \to \nu_e}$ as a function of $\Delta m^2/A$ and $\sin^2\theta$.

a survival probability of approximately unity elsewhere. The larger value of θ (dashed line) is in the adiabatic region but close to the limit. This results in mixing of the two generations not only on the inside of the triangle but also on the outside. This explains why the survival probability for the large value of θ is significantly lower than unity outside the triangle compared to the case of the lower mixing angle.

Comparing the steepness of the dips in the curves in Figure 2 with the color gradient in Figure 3 it suddenly makes sense why the lines that constitute the MSW triangle in [FY] (Figure (8.10)) are only approximate since the steepness of the dips evens out for large θ. This is evident from looking at $\sin^2 2\theta = 1$ where there is no dip at all. This is just as expected since $\sin^2 2\theta = 1$ corresponds to a mixing angle of $\pi/4$ which is the angle of maximal mixing between the two generations.